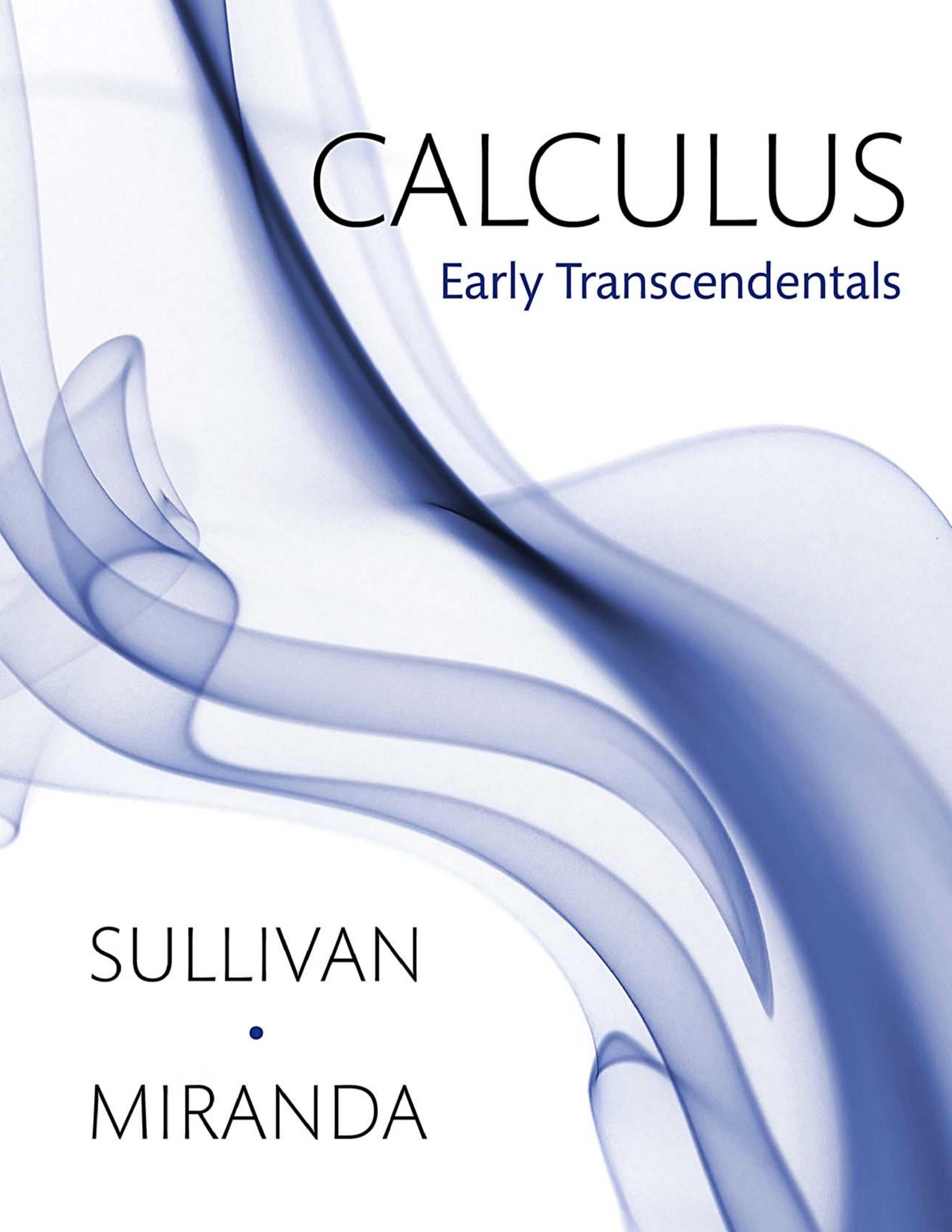
CALCULUS

Early Transcendentals



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Calculus—What Is It?

Calculus is a part of mathematics that evolved much later than other subjects. Algebra, geometry, and trigonometry were developed in ancient times, but calculus as we know it did not appear until the seventeenth century.

The first evidence of calculus has its roots in ancient mathematics. For example, in his book, *A History of* π, Petr Beckmann explains that Greek mathematician Archimedes (287–212 BCE) "took the step from the concept of 'equal to' to the concept of 'arbitrarily close to' or 'as closely as desired'... and thus reached the threshold of the differential calculus, just as his method of squaring the parabola reached the threshold of the integral calculus."* But it was not until Sir Isaac Newton and Gottfried Wilhelm Leibniz, each working independently, expanded, organized, and applied these early ideas, that the subject we now know as calculus was born.

Although we attribute the birth of calculus to Newton and Leibniz, many other mathematicians, particularly those in the eighteenth and nineteenth centuries, contributed greatly to the body and rigor of calculus. You will encounter many of their names and contributions as you pursue your study of calculus.

But, what is calculus? Why is it given such notoriety?

The simple answer is: calculus models change. Since the world and most things in it are constantly changing, mathematics that explains change becomes immensely useful.

Calculus has two major branches, differential calculus and integral calculus. Let's take a peek at what calculus is by looking at two problems that prompted the development of calculus.

The Tangent Problem—The Basis of Differential Calculus

Suppose we want to find the slope of the line tangent to the graph of a function at some point $P = (x_1, y_1)$. See Figure 1(a). Since the tangent line necessarily contains the point P, it remains only to find the slope to identify the tangent line. Suppose we repeatedly zoom in on the graph of the function at the point P. See Figure 1(b). If we can zoom in close enough, then the graph of the function will look approximately linear, and we can choose a point Q, on the graph of the function different from the point P, and use the formula for slope.

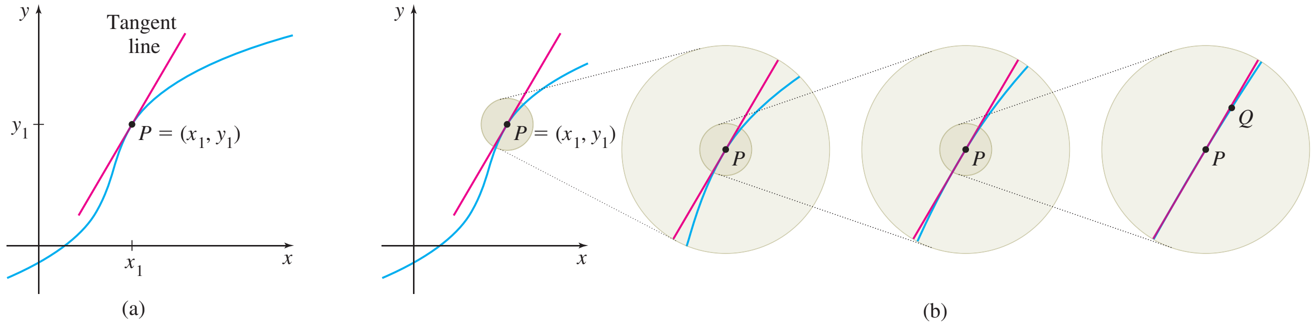

Figure 1

Repeatedly zooming in on the point P is equivalent to choosing a point Q closer and closer to the point P. Notice that as we zoom in on P, the line connecting the points P and Q, called a secant line, begins to look more and more like the tangent line to the graph of the function at the point P. If the point Q can be made as close as we please to the point P, without equaling the point P, then the slope of the tangent line $m_{\tan}$ can be found. This formulation leads to differential calculus, the study of the derivative of a function.

The derivative gives us information about how a function changes at a given instant and can be used to solve problems involving velocity and acceleration; marginal cost and profit; and the rate of change of a chemical reaction. Derivatives are the subjects of Chapters 2 through 4.

The Area Problem—The Basis of Integral Calculus

If we want to find the area of a rectangle or the area of a circle, formulas are available. (see Figure 2). But what if the figure is curvy, but not circular as in Figure 3? How do we find this area?

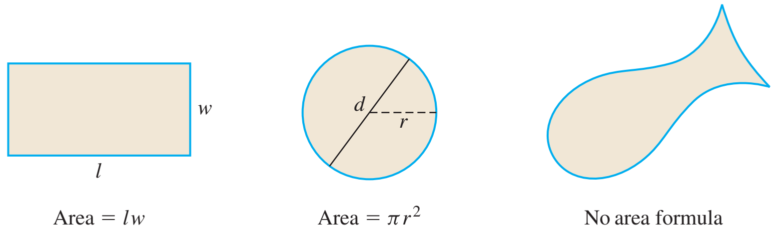

Figure 2 **Figure 3**

Calculus provides a way. Look at Figure 4(a). It shows the graph of $y = x^2$ from $x = 0$ to $x = 1$. Suppose we want to find the shaded area.

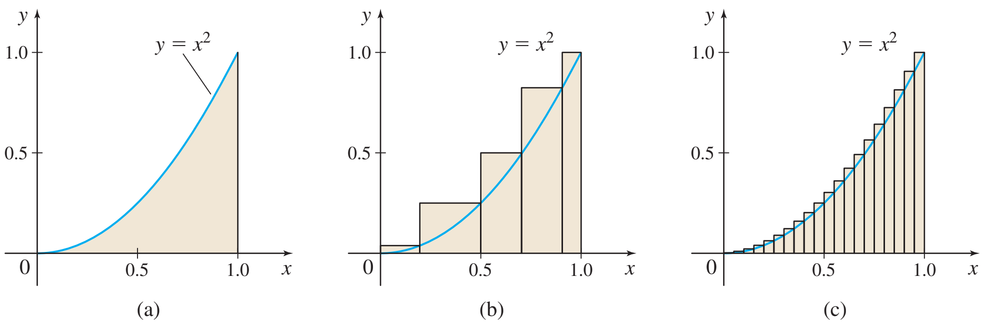

Figure 4

By subdividing the x-axis from 0 to 1 into small segments and drawing a rectangle of height x^2 above each segment, as in Figure 4(b), we can find the area of each rectangle and add them together. This sum approximates the shaded area in Figure 4(a). The smaller we make the segments of the x-axis and the more rectangles we draw, the better the approximation becomes. See Figure 4(c). This formulation leads to integral calculus, and the study of the integral of a function.

Two Problems—One Subject?

At first, differential calculus (the tangent problem) and integral calculus (the area problem) appear to be different, so why call both of them calculus? The Fundamental Theorem of Calculus establishes that the derivative and the integral are related. In fact, one of Newton's teachers, Isaac Barrow, recognized that the tangent problem and the area problem are closely related, and that derivatives and integrals are inverses of each other. Both Newton and Leibniz formalized this relationship between derivatives and integrals in the *Fundamental Theorem of Calculus.*

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* Beckmann, P. (1976). *A History of* π (3rd. ed., p. 64). New York: St. Martin's Press.

TABLE OF DERIVATIVES

1. $\dfrac{d}{dx}c=0,\quad c$ is a constant

2. $\dfrac{d}{dx}(u\pm v)=\dfrac{du}{dx}\pm\dfrac{dv}{dx}$

3. $\dfrac{d}{dx}(cu)=c\dfrac{du}{dx},\quad c$ a constant

4. $\dfrac{d}{dx}(uv)=u\dfrac{dv}{dx}+v\dfrac{du}{dx}$

5. $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)=\dfrac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}$

6. $\dfrac{dy}{dx}=\dfrac{dy}{du}\dfrac{du}{dx},\quad y=f(u),\quad u=g(x)$

7. $\dfrac{d}{dx}x^a=ax^{a-1}$

8. $\dfrac{d}{dx}\sin x=\cos x$

9. $\dfrac{d}{dx}\cos x=-\sin x$

10. $\dfrac{d}{dx}\tan x=\sec^2 x$

11. $\dfrac{d}{dx}\cot x=-\csc^2 x$

12. $\dfrac{d}{dx}\sec x=\sec x\tan x$

13. $\dfrac{d}{dx}\csc x=-\csc x\cot x$

14. $\dfrac{d}{dx}\ln x=\dfrac{1}{x}$

15. $\dfrac{d}{dx}e^x=e^x$

16. $\dfrac{d}{dx}a^x=a^x\ln a,\quad a>0,\ a\neq 1$

17. $\dfrac{d}{dx}\log_a x=\dfrac{1}{x\ln a},\quad a>0,\ a\neq 1$

18. $\dfrac{d}{dx}\sin^{-1}x=\dfrac{1}{\sqrt{1-x^2}}$

19. $\dfrac{d}{dx}\tan^{-1}x=\dfrac{1}{1+x^2}$

20. $\dfrac{d}{dx}\sec^{-1}x=\dfrac{1}{x\sqrt{x^2-1}}$

21. $\dfrac{d}{dx}\cos^{-1}x=-\dfrac{1}{\sqrt{1-x^2}}$

22. $\dfrac{d}{dx}\cot^{-1}x=-\dfrac{1}{1+x^2}$

23. $\dfrac{d}{dx}\csc^{-1}x=-\dfrac{1}{x\sqrt{x^2-1}}$

24. $\dfrac{d}{dx}\sinh x=\cosh x$

25. $\dfrac{d}{dx}\cosh x=\sinh x$

26. $\dfrac{d}{dx}\tanh x=\text{sech}^2 x$

27. $\dfrac{d}{dx}\coth x=-\text{csch}^2 x$

28. $\dfrac{d}{dx}\text{sech}\, x=-\text{sech}\, x\tanh x$

29. $\dfrac{d}{dx}\text{csch}\, x=-\text{csch}\, x\coth x$

30. $\dfrac{d}{dx}\sinh^{-1}x=\dfrac{1}{\sqrt{1+x^2}}$

31. $\dfrac{d}{dx}\cosh^{-1}x=\dfrac{1}{\sqrt{x^2-1}}$

32. $\dfrac{d}{dx}\tanh^{-1}x=\dfrac{1}{1-x^2}$

TABLE OF INTEGRALS

General Formulas

1. $\int[f(x)\pm g(x)]\,dx=\int f(x)\,dx\pm\int g(x)\,dx$

2. $\int kf(x)\,dx=k\int f(x)\,dx,\quad k$ a constant

3. $\int u\,dv=uv-\int v\,du$

Essential Integrals

1. $\int x^a\,dx=\dfrac{x^{a+1}}{a+1}+C,\quad a\neq -1$

2. $\int\dfrac{1}{x}\,dx=\ln|x|+C$

3. $\int e^x\,dx=e^x+C$

4. $\int\sin x\,dx=-\cos x+C$

5. $\int\cos x\,dx=\sin x+C$

6. $\int\tan x\,dx=\ln|\sec x|+C$

7. $\int\sec x\,dx=\ln|\sec x+\tan x|+C$

8. $\int\sec^2 x\,dx=\tan x+C$

9. $\int\sec x\tan x\,dx=\sec x+C$

10. $\int\dfrac{dx}{\sqrt{1-x^2}}=\sin^{-1}x+C$

11. $\int\dfrac{dx}{1+x^2}=\tan^{-1}x+C$

12. $\int\dfrac{dx}{x\sqrt{x^2-1}}=\sec^{-1}x+C$

13. $\int\cot x\,dx=\ln|\sin x|+C$

14. $\int\csc x\,dx=\ln|\csc x-\cot x|+C$

15. $\int\csc^2 x\,dx=-\cot x+C$

16. $\int\csc x\cot x\,dx=-\csc x+C$

17. $\int\dfrac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\dfrac{x}{a}+C,\quad a>0$

18. $\int\dfrac{dx}{a^2+x^2}=\dfrac{1}{a}\tan^{-1}\dfrac{x}{a}+C,\quad a>0$

19. $\int\dfrac{dx}{x\sqrt{x^2-a^2}}=\dfrac{1}{a}\sec^{-1}\dfrac{x}{a}+C,\quad a>0$

20. $\int a^x\,dx=\dfrac{a^x}{\ln a}+C,\quad a>0,\quad a\neq 1$

Useful Integrals

21. $\int \sinh x\, dx = \cosh x + C$

22. $\int \cosh x\, dx = \sinh x + C$

23. $\int \operatorname{sech}^2 x\, dx = \tanh x + C$

24. $\int \operatorname{csch}^2 x\, dx = -\coth x + C$

25. $\int \operatorname{sech} x \tanh x\, dx = -\operatorname{sech} x + C$

26. $\int \operatorname{csch} x \coth x\, dx = -\operatorname{csch} x + C$

27. $\int \frac{dx}{a+bx} = \frac{1}{b}\ln|a+bx| + C$

28. $\int \frac{x\,dx}{a+bx} = \frac{1}{b^2}(a+bx - a\ln|a+bx|) + C$

29. $\int \frac{x\,dx}{(a+bx)^2} = \frac{a}{b^2(a+bx)} + \frac{1}{b^2}\ln|a+bx| + C$

30. $\int \frac{x^2\,dx}{(a+bx)^2} = \frac{1}{b^3}\left(a+bx - \frac{a^2}{a+bx} - 2a\ln|a+bx|\right) + C$

31. $\int \frac{dx}{x(a+bx)^2} = \frac{1}{a(a+bx)} - \frac{1}{a^2}\ln\left|\frac{a+bx}{x}\right| + C$

32. $\int \frac{dx}{x^2(a+bx)} = -\frac{1}{ax} + \frac{b}{a^2}\ln\left|\frac{a+bx}{x}\right| + C$

33. $\int x(a+bx)^n\,dx = \frac{(a+bx)^{n+1}}{b^2}\left(\frac{a+bx}{n+2} - \frac{a}{n+1}\right) + C, \quad n \neq -1, -2$

34. $\int \frac{x\,dx}{(a+bx)(c+dx)} = \frac{1}{bc-ad}\left(-\frac{a}{b}\ln|a+bx| + \frac{c}{d}\ln|c+dx|\right) + C, \quad bc-ad \neq 0$

35. $\int \frac{x\,dx}{(a+bx)^2(c+dx)} = \frac{1}{bc-ad}\left[\frac{a}{b(a+bx)} + \frac{c}{bc-ad}\ln\left|\frac{a+bx}{c+dx}\right|\right] + C, \quad bc-ad \neq 0$

36. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right| + C$

37. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$

38. $\int \frac{dx}{(a^2 \pm x^2)^n} = \frac{1}{2(n-1)a^2}\left[\frac{x}{(a^2 \pm x^2)^{n-1}} + (2n-3)\int \frac{dx}{(a^2 \pm x^2)^{n-1}}\right], \quad n \neq 1$

39. $\int \frac{dx}{(x^2-a^2)^n} = \frac{1}{2(n-1)a^2}\left[-\frac{x}{(x^2-a^2)^{n-1}} - (2n-3)\int \frac{dx}{(x^2-a^2)^{n-1}}\right], \quad n \neq 1$

Integrals Containing $\sqrt{a+bx}$

40. $\displaystyle\int x\sqrt{a+bx}\,dx = \frac{2}{15b^2}(3bx-2a)(a+bx)^{3/2}+C$

41. $\displaystyle\int x^n\sqrt{a+bx}\,dx = \frac{2}{b(2n+3)}\left[x^n(a+bx)^{3/2} - na\int x^{n-1}\sqrt{a+bx}\,dx\right]$

42. $\displaystyle\int \frac{x\,dx}{\sqrt{a+bx}} = \frac{2}{3b^2}(bx-2a)\sqrt{a+bx}+C$

43. $\displaystyle\int \frac{x^2dx}{\sqrt{a+bx}} = \frac{2}{15b^2}(8a^2-4abx+3b^2x^2)\sqrt{a+bx}+C$

44. $\displaystyle\int \frac{x^n dx}{\sqrt{a+bx}} = \frac{2x^n\sqrt{a+bx}}{b(2n+1)} - \frac{2na}{b(2n+1)}\int\frac{x^{n-1}dx}{\sqrt{a+bx}}$

45. $\displaystyle\int \frac{dx}{x\sqrt{a+bx}} = \begin{cases} \dfrac{1}{\sqrt{a}}\ln\left|\dfrac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}}\right|+C, & a>0 \\[2ex] \dfrac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\dfrac{a+bx}{-a}}+C, & a<0 \end{cases}$

46. $\displaystyle\int \frac{dx}{x^n\sqrt{a+bx}} = -\frac{\sqrt{a+bx}}{a(n-1)x^{n-1}} - \frac{b(2n-3)}{2a(n-1)}\int\frac{dx}{x^{n-1}\sqrt{a+bx}}$

47. $\displaystyle\int \frac{\sqrt{a+bx}}{x}dx = 2\sqrt{a+bx}+a\int\frac{dx}{x\sqrt{a+bx}}$

48. $\displaystyle\int \frac{\sqrt{a+bx}}{x^2}dx = -\frac{\sqrt{a+bx}}{x}+\frac{b}{2}\int\frac{dx}{x\sqrt{a+bx}}$

Integrals Containing $\sqrt{x^2 \pm a^2}$

49. $\displaystyle\int \sqrt{x^2\pm a^2}\,dx = \frac{x}{2}\sqrt{x^2\pm a^2} \pm \frac{a^2}{2}\ln\left|x+\sqrt{x^2\pm a^2}\right|+C$

50. $\displaystyle\int x\sqrt{x^2\pm a^2}\,dx = \frac{1}{3}(x^2\pm a^2)^{3/2}+C$

51. $\displaystyle\int x^2\sqrt{x^2\pm a^2}\,dx = \frac{x}{8}(2x^2\pm a^2)\sqrt{x^2\pm a^2} - \frac{a^4}{8}\ln\left|x+\sqrt{x^2\pm a^2}\right|+C$

52. $\displaystyle\int \frac{\sqrt{x^2+a^2}}{x}dx = \sqrt{x^2+a^2} - a\ln\left|\frac{a+\sqrt{x^2+a^2}}{x}\right|+C$

53. $\displaystyle\int \frac{\sqrt{x^2-a^2}}{x}dx = \sqrt{x^2-a^2} - a\sec^{-1}\frac{x}{a}+C$

54. $\displaystyle\int \frac{\sqrt{x^2\pm a^2}}{x^2}dx = -\frac{\sqrt{x^2\pm a^2}}{x} + \ln\left|x+\sqrt{x^2\pm a^2}\right|+C$

55. $\displaystyle\int \frac{dx}{\sqrt{x^2\pm a^2}} = \ln\left|x+\sqrt{x^2\pm a^2}\right|+C$

56. $\displaystyle\int \frac{x^2dx}{\sqrt{x^2\pm a^2}} = \frac{x}{2}\sqrt{x^2\pm a^2} \pm \frac{a^2}{2}\ln\left|x+\sqrt{x^2\pm a^2}\right|+C$

57. $\displaystyle\int \frac{dx}{x\sqrt{x^2+a^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{x^2+a^2}}{x}\right|+C$

58. $\displaystyle\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a}+C$

59. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2\pm a^2}} = \mp\frac{\sqrt{x^2\pm a^2}}{a^2x}+C$

60. $\displaystyle\int (x^2\pm a^2)^{3/2}dx = \frac{x}{8}(2x^2\pm 5a^2)\sqrt{x^2\pm a^2} + \frac{3a^4}{8}\ln\left|x+\sqrt{x^2\pm a^2}\right|+C$

61. $\displaystyle\int \frac{dx}{(x^2\pm a^2)^{3/2}} = \pm\frac{x}{a^2\sqrt{x^2\pm a^2}}+C$

Integrals Containing $\sqrt{a^2 - x^2}$

62. $\displaystyle\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

63. $\displaystyle\int x^2\sqrt{a^2-x^2}\,dx = \frac{x}{8}(2x^2-a^2)\sqrt{a^2-x^2} + \frac{a^4}{8}\sin^{-1}\frac{x}{a} + C$

64. $\displaystyle\int \frac{\sqrt{a^2-x^2}}{x}\,dx = \sqrt{a^2-x^2} - a\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right| + C$

65. $\displaystyle\int \frac{\sqrt{a^2-x^2}}{x^2}\,dx = -\frac{\sqrt{a^2-x^2}}{x} - \sin^{-1}\frac{x}{a} + C$

66. $\displaystyle\int \frac{x^2}{\sqrt{a^2-x^2}}\,dx = -\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

67. $\displaystyle\int \frac{dx}{x\sqrt{a^2-x^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right| + C$

68. $\displaystyle\int \frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2x} + C$

69. $\displaystyle\int (a^2-x^2)^{3/2}dx = \frac{x}{4}(a^2-x^2)^{3/2} + \frac{3a^2x}{8}\sqrt{a^2-x^2} + \frac{3a^4}{8}\sin^{-1}\frac{x}{a} + C$

70. $\displaystyle\int \frac{dx}{(a^2-x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2-x^2}} + C$

Integrals Containing $\sqrt{2ax - x^2}$

71. $\displaystyle\int \sqrt{2ax-x^2}\,dx = \frac{x-a}{2}\sqrt{2ax-x^2} + \frac{a^2}{2}\cos^{-1}\left(\frac{a-x}{a}\right) + C$

72. $\displaystyle\int x\sqrt{2ax-x^2}\,dx = \frac{2x^2-ax-3a^2}{6}\sqrt{2ax-x^2} + \frac{a^3}{2}\cos^{-1}\left(\frac{a-x}{a}\right) + C$

73. $\displaystyle\int \frac{\sqrt{2ax-x^2}}{x}\,dx = \sqrt{2ax-x^2} + a\cos^{-1}\left(\frac{a-x}{a}\right) + C$

74. $\displaystyle\int \frac{\sqrt{2ax-x^2}}{x^2}\,dx = -\frac{2\sqrt{2ax-x^2}}{x} - \cos^{-1}\left(\frac{a-x}{a}\right) + C$

75. $\displaystyle\int \frac{dx}{\sqrt{2ax-x^2}} = \cos^{-1}\left(\frac{a-x}{a}\right) + C$

76. $\displaystyle\int \frac{x\,dx}{\sqrt{2ax-x^2}} = -\sqrt{2ax-x^2} + a\cos^{-1}\left(\frac{a-x}{a}\right) + C$

77. $\displaystyle\int \frac{x^2\,dx}{\sqrt{2ax-x^2}} = -\frac{x+3a}{2}\sqrt{2ax-x^2} + \frac{3a^2}{2}\cos^{-1}\left(\frac{a-x}{a}\right) + C$

78. $\displaystyle\int \frac{dx}{x\sqrt{2ax-x^2}} = -\frac{\sqrt{2ax-x^2}}{ax} + C$

79. $\displaystyle\int \frac{\sqrt{2ax-x^2}}{x^n}\,dx = \frac{(2ax-x^2)^{3/2}}{(3-2n)ax^n} + \frac{n-3}{(2n-3)a}\int \frac{\sqrt{2ax-x^2}}{x^{n-1}}\,dx, \quad n \neq \frac{3}{2}$

80. $\displaystyle\int \frac{x^n\,dx}{\sqrt{2ax-x^2}} = -\frac{x^{n-1}\sqrt{2ax-x^2}}{n} + \frac{a(2n-1)}{n}\int \frac{x^{n-1}}{\sqrt{2ax-x^2}}\,dx$

81. $\displaystyle\int \frac{dx}{x^n\sqrt{2ax-x^2}} = \frac{\sqrt{2ax-x^2}}{a(1-2n)x^n} + \frac{n-1}{(2n-1)a}\int \frac{dx}{x^{n-1}\sqrt{2ax-x^2}}$

82. $\displaystyle\int \frac{dx}{(2ax-x^2)^{3/2}} = \frac{x-a}{a^2\sqrt{2ax-x^2}} + C$

83. $\displaystyle\int \frac{x\,dx}{(2ax-x^2)^{3/2}} = \frac{x}{a\sqrt{2ax-x^2}} + C$

Integrals Containing Trigonometric Functions

84. $\int \sin^2 x\, dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$

85. $\int \cos^2 x\, dx = \frac{x}{2} + \frac{\sin 2x}{4} + C$

86. $\int \tan^2 x\, dx = \tan x - x + C$

87. $\int \cot^2 x\, dx = -\cot x - x + C$

88. $\int \sec^3 x\, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$

89. $\int \csc^3 x\, dx = -\frac{1}{2}\csc x \cot x + \frac{1}{2}\ln|\csc x - \cot x| + C$

90. $\int \sin^n x\, dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x\, dx$

91. $\int \cos^n x\, dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\, dx$

92. $\int \tan^n x\, dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\, dx$

93. $\int \cot^n x\, dx = \frac{-1}{n-1}\cot^{n-1} x - \int \cot^{n-2} x\, dx$

94. $\int \sec^n x\, dx = \frac{1}{n-1}\tan x \sec^{n-2} x + \frac{n-2}{n-1}\int \sec^{n-2} x\, dx$

95. $\int \csc^n x\, dx = \frac{-1}{n-1}\cot x \csc^{n-2} x + \frac{n-2}{n-1}\int \csc^{n-2} x\, dx$

96. $\int \sin mx \sin nx\, dx = -\frac{\sin(m+n)x}{2(m+n)} + \frac{\sin(m-n)x}{2(m-n)} + C, \quad m^2 \neq n^2$

97. $\int \cos mx \cos nx\, dx = \frac{\sin(m+n)x}{2(m+n)} + \frac{\sin(m-n)x}{2(m-n)} + C, m^2 \neq n^2$

98. $\int \sin mx \cos nx\, dx = -\frac{\cos(m+n)x}{2(m+n)} - \frac{\cos(m-n)x}{2(m-n)} + C, \quad m^2 \neq n^2$

99. $\int x \sin x\, dx = \sin x - x\cos x + C$

100. $\int x \cos x\, dx = \cos x + x \sin x + C$

101. $\int x^2 \sin x\, dx = 2x \sin x + (2 - x^2)\cos x + C$

102. $\int x^2 \cos x\, dx = 2x \cos x + (x^2 - 2)\sin x + C$

103. $\int x^n \sin x\, dx = -x^n \cos x + n\int x^{n-1} \cos x\, dx$

104. $\int x^n \cos x\, dx = x^n \sin x - n\int x^{n-1} \sin x\, dx$

105. **(a)** $\int \sin^m x \cos^n x\, dx = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2} x \cos^n x\, dx$

(b) $\int \sin^m x \cos^n x\, dx = -\frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n}\int \sin^m x \cos^{n-2} x\, dx$

(if $m = -n$ use formula 92 or 93.)

Integrals Containing Inverse Trigonometric Functions

106. $\int \sin^{-1} x\, dx = x \sin^{-1} x + \sqrt{1-x^2} + C$

107. $\int \cos^{-1} x\, dx = x \cos^{-1} x - \sqrt{1-x^2} + C$

108. $\int \tan^{-1} x\, dx = x \tan^{-1} x - \frac{1}{2}\ln(1+x^2) + C$

109. $\int x \sin^{-1} x\, dx = \frac{2x^2-1}{4}\sin^{-1} x + \frac{x\sqrt{1-x^2}}{4} + C$

110. $\int x \cos^{-1} x\, dx = \frac{2x^2-1}{4}\cos^{-1} x - \frac{x\sqrt{1-x^2}}{4} + C$

111. $\int x \tan^{-1} x\, dx = \frac{x^2+1}{2}\tan^{-1} x - \frac{x}{2} + C$

112. $\int x^n \sin^{-1} x\, dx = \frac{1}{n+1}\left(x^{n+1}\sin^{-1} x - \int \frac{x^{n+1}dx}{\sqrt{1-x^2}}\right), \quad n \neq -1$

113. $\int x^n \cos^{-1} x\, dx = \frac{1}{n+1}\left(x^{n+1}\cos^{-1} x + \int \frac{x^{n+1}dx}{\sqrt{1-x^2}}\right), \quad n \neq -1$

114. $\int x^n \tan^{-1} x\, dx = \frac{1}{n+1}\left(x^{n+1}\tan^{-1} x - \int \frac{x^{n+1}dx}{1+x^2}\right), \quad n \neq -1$

Integrals Containing Exponential and Logarithmic Functions

115. $\int xe^{ax}\, dx = \frac{1}{a^2}(ax-1)e^{ax} + C$

116. $\int x^n e^{ax}\, dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}\, dx$

117. $\int \frac{e^x}{x^n}dx = -\frac{e^x}{(n-1)x^{n-1}} + \frac{1}{n-1}\int \frac{e^x}{x^{n-1}}dx$

118. $\int \ln x\, dx = x\ln x - x + C$

119. $\int (\ln x)^n\, dx = x(\ln x)^n - n\int (\ln x)^{n-1}\, dx$

120. $\int x^n \ln x\, dx = \left(\frac{x^{n+1}}{n+1}\right)\left(\ln x - \frac{1}{n+1}\right) + C$

121. $\int \frac{(\ln x)^n}{x}dx = \frac{(\ln x)^{n+1}}{n+1} + C, \quad n \neq -1$

122. $\int \frac{dx}{x\ln x} = \ln|\ln x| + C$

123. $\int x^m (\ln x)^n\, dx = \frac{x^{m+1}(\ln x)^n}{m+1} - \frac{n}{m+1}\int x^m (\ln x)^{n-1}\, dx$

124. $\int \frac{x^m}{(\ln x)^n}dx = -\frac{x^{m+1}}{(n-1)(\ln x)^{n-1}} + \frac{m+1}{n-1}\int \frac{x^m}{(\ln x)^{n-1}}dx$

125. $\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C$

126. $\int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + C$

Integrals Containing Hyperbolic Functions

127. $\int \sinh x\, dx = \cosh x + C$

128. $\int \cosh x\, dx = \sinh x + C$

129. $\int \tanh x\, dx = \ln\cosh x + C$

130. $\int \coth x\, dx = \ln|\sinh x| + C$

131. $\int \operatorname{sech} x\, dx = \tan^{-1}(\sinh x) + C$

132. $\int \operatorname{csch} x\, dx = \ln\left|\tanh\frac{x}{2}\right| + C$

133. $\int \operatorname{sech}^2 x\, dx = \tanh x + C$

134. $\int \operatorname{csch}^2 x\, dx = -\coth x + C$

135. $\int \operatorname{sech} x \tanh x\, dx = -\operatorname{sech} x + C$

136. $\int \operatorname{csch} x \coth x\, dx = -\operatorname{csch} x + C$

137. $\int \sinh^2 x\, dx = \frac{\sinh 2x}{4} - \frac{x}{2} + C$

138. $\int \cosh^2 x\, dx = \frac{\sinh 2x}{4} + \frac{x}{2} + C$

139. $\int \tanh^2 x\, dx = x - \tanh x + C$

140. $\int \coth^2 x\, dx = x - \coth x + C$

141. $\int x\sinh x\, dx = x\cosh x - \sinh x + C$

142. $\int x\cosh x\, dx = x\sinh x - \cosh x + C$

143. $\int x^n \sinh x\, dx = x^n\cosh x - n\int x^{n-1}\cosh x\, dx$

144. $\int x^n \cosh x\, dx = x^n\sinh x - n\int x^{n-1}\sinh x\, dx$

CALCULUS

Early Transcendentals

MICHAEL SULLIVAN
Chicago State University

• • •

KATHLEEN MIRANDA
State University of New York,
Old Westbury

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Michael Sullivan

Michael Sullivan, Emeritus Professor of Mathematics at Chicago State University, received a PhD in mathematics from the Illinois Institute of Technology. Before retiring, Mike taught at Chicago State for 35 years, where he honed an approach to teaching and writing that forms the foundation for his textbooks. Mike has been writing for over 35 years and currently has 15 books in print. His books have been awarded both Texty and McGuffey awards from TAA.

Mike is a member of the American Mathematical Association of Two Year Colleges, the American Mathematical Society, the Mathematical Association of America, and the Text and Academic Authors Association (TAA), and has received the TAA Lifetime Achievement Award in 2007. His influence in the field of mathematics extends to his four children: Kathleen, who teaches college mathematics; Michael III, who also teaches college mathematics, and who is his coauthor on two precalculus series; Dan, who is a sales director for a college textbook publishing company; and Colleen, who teaches middle-school and secondary school mathematics. Twelve grandchildren round out the family.

Mike would like to dedicate this text to his four children, 12 grandchildren, and future generations.

Kathleen Miranda

Kathleen Miranda, Ed.D. from St. John's University, is an Emeritus Associate Professor of the State University of New York (SUNY) where she taught for 25 years. Kathleen is a recipient of the prestigious New York State Chancellor's Award for Excellence in Teaching, and particularly enjoys teaching mathematics to underprepared and fearful students. In addition to her extensive classroom experience, Kathleen has worked as accuracy reviewer and solutions author on several mathematics textbooks, including Michael Sullivan's *Brief Calculus* and *Finite Mathematics.* Kathleen's goal is to help students unlock the complexities of calculus and appreciate its many applications.

Kathleen has four children: Edward, a plastic surgeon in San Francisco; James, an emergency medicine physician in Philadelphia; Kathleen, a chemical engineer working on vaccines; and Michael, a management consultant specializing in corporate strategy.

Kathleen would like to dedicate this text to her children and grandchildren.

Preface

The challenges facing instructors of calculus are daunting. Diversity among students, both in their mathematical preparedness for learning calculus and in their ultimate educational and career goals, is vast and growing. There is just not enough classroom time to teach every topic in the syllabus, to answer every student's questions, or to delve into the rich examples and applications that showcase the beauty and utility of calculus. As mathematics instructors we share these frustrations with you. As authors our goal is to create a student-oriented textbook that supports your teaching philosophy and promotes student success and confidence.

Promoting Student Success Is the Central Theme

Our goal is to write a mathematically precise calculus book that embraces proven pedagogical features to increase both student and instructor success. Many of these features are structural, but there are also many less obvious, intrinsic features embedded in the text.

- **The text is written to be read by students.** The language is simple, clear, and consistent. Definitions are simply stated and consistently used. Theorems are given names where appropriate. Numbering of definitions, equations, and theorems is kept to a minimum.

  The careful use of color throughout the text brings attention to important statements such as definitions and theorems. Important formulas are boxed, and procedures and summaries are called out so that a student can quickly look back and review the main points of the section.

- **The text is written to prepare students.** Whether students have educational goals that end in the social sciences, in the life and/or physical sciences, in engineering, or with a PhD in mathematics, the text provides ample practice, applications, and the mathematical precision and rigor required to prepare students to pursue their goals.

- **The text is written to be mastered by students.** Carefully used pedagogical features are found throughout the text. These features provide structure and form a carefully crafted learning system that helps students get the most out of their study. From our experience, students who use the features are more successful in calculus.

Pedagogical Features Promote Student Success

Just In Time Review Students forget; they often do not make connections. Instructors lament, "The students are not prepared!" So, throughout the text there are margin notes labeled **NEED TO REVIEW?** followed by a topic and page references. The **NEED TO REVIEW?** reference points to the discussion of a concept used in the current presentation.

NEED TO REVIEW? Summation notation is discussed in Appendix A.5, pp. A-38 to A-43.

The sum s_n of the areas of the n rectangles approximates the area A. That is,

$$A \approx s_n = f(c_1)\Delta x + f(c_2)\Delta x + \cdots + f(c_i)\Delta x + \cdots + f(c_n)\Delta x = \sum_{i=1}^{n} f(c_i)\Delta x$$

RECALL margin notes provide a quick refresher of key results that are being used in theorems, definitions, and examples.

RECALL $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Using summation properties, we get

$$S_n = \sum_{i=1}^{n} \frac{300}{n^2} i = \frac{300}{n^2} \sum_{i=1}^{n} i = \frac{300}{n^2} \frac{n(n+1)}{2} = 150\left(\frac{n+1}{n}\right) = 150\left(1 + \frac{1}{n}\right)$$

Additional margin notes are included throughout the text to help students understand and engage with the concepts. **IN WORDS** notes translate complex formulas, theorems, proofs, rules, and definitions using plain language that provide students with an alternate way to learn the concepts.

IN WORDS The average value $\bar{y} = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$ of a function f equals the value $f(u)$ in the Mean Value Theorem for Integrals.

DEFINITION Average Value of a Function over an Interval

Let f be a function that is continuous on the closed interval $[a, b]$. The **average value $\bar{y}$ of f over $[a, b]$** is

$$\boxed{\bar{y} = \frac{1}{b-a}\int_a^b f(x)\,dx} \qquad (8)$$

CAUTIONS and **NOTES** provide supporting details or warnings about common pitfalls. **ORIGINS** give biographical information or historical details about key figures and discoveries in the development of calculus.

CAUTION In writing an indefinite integral $\int f(x)\,dx$, remember to include the "dx."

ORIGINS Willard F. Libby (1908–1980) grew up in California and went to college and graduate school at UC Berkeley. Libby was a physical chemist who taught at the University of Chicago and later at UCLA. While at Chicago, he developed the methods for using natural carbon-14 to date archaeological artifacts. Libby won the Nobel Prize in Chemistry in 1960 for this work.

Learning Objectives Students often need focus. Each section begins with a set of **Objectives** that serve as a broad outline of the section. The objectives help students study effectively by focusing attention on the concepts being covered. Each learning objective is supported by appropriate definitions, theorems, and proofs. One or more carefully chosen examples enhance the learning objective, and where appropriate, an application example is also included.

Learning objectives help instructors prepare a syllabus that includes the important topics of calculus, and concentrate instruction on mastery of these topics. They are also helpful for answering the familiar question, "What is on the test?"

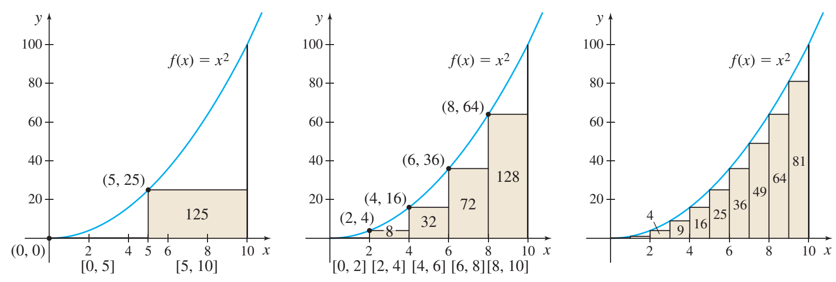
5.1 Area

OBJECTIVES *When you finish this section, you should be able to:*

1. Approximate the area under the graph of a function (p. 2)
2. Find the area under the graph of a function (p. 6)

Effective Use of Color The text contains an abundance of graphs and illustrations that carefully utilize color to make concepts easier to visualize.

Dynamic Figures The text includes many pieces of art that students, can interact with through the online e-Book. These dynamic figures, indicated by the icon DF next to the figure label, illustrate select principles of calculus including limits, rates of change, solids of revolution, convergence, and divergence.



(a) $s_2 = 125$ sq. units (b) $s_5 = 240$ sq. units (c) $s_{10} = 285$ sq. units

Figure 9 DF

EXAMPLE 6 Using L'Hôpital's Rule to Find a One-Sided Limit

Find $\lim\limits_{x\to 0^+} \dfrac{\cot x}{\ln x}$.

$$\lim_{x\to 0^+}\frac{\cot x}{\ln x} \underset{\uparrow}{=} \lim_{x\to 0^+}\frac{\frac{d}{dx}\cot x}{\frac{d}{dx}\ln x} = \lim_{x\to 0^+}\frac{-\csc^2 x}{\frac{1}{x}} \underset{\uparrow}{=} -\lim_{x\to 0^+}\frac{x}{\sin^2 x} \underset{\uparrow}{=} -\lim_{x\to 0^+}\frac{\frac{d}{dx}x}{\frac{d}{dx}\sin^2 x}$$

L'Hôpital's Rule $\quad \csc^2 x = \dfrac{1}{\sin^2 x} \quad$ L'Hôpital's Rule

Examples with Detailed and Annotated Solutions: Examples are named according to their purpose. The text includes hundreds of examples with detailed step-by-step solutions. Additional annotations provide students with the formula used, or the reasoning needed, to perform a particular step of the solution. Where procedural steps for solving a type of problem are given in the text, these steps are followed in the solved examples. Often a figure or a graph is included to complement the solution.

CHAPTER 5 PROJECT Managing the Klamath River

There is a gauge on the Klamath River, just downstream from the dam at Keno, Oregon. The U.S. Geological Survey has posted flow rates for this gauge every month since 1930. The averages of these monthly measurements since 1930 are given in Table 2. Notice that the data in Table 2 measure the rate of change of the volume V in cubic feet of water each second over one year; that is, the table gives $\dfrac{dV}{dt} = V'(t)$ in cubic feet per second, where t is in months.

Application Examples and Chapter Projects The inclusion of application examples motivates and underscores the conceptual theory, and powerfully conveys the message that calculus is a beneficial and a relevant subject to master. It is this combination of concepts with applied examples and exercises that gives students the tools not only to complete the exercises successfully, but also to comprehend the mathematics behind them.

This message is further reinforced by the case studies that open each chapter, demonstrating how major concepts apply to recognizable and often contemporary situations in biology, environmental studies, astronomy, engineering, technology, and other fields. Students see this case study again at the end of the chapter, where an extended project presents questions that guide students through a solution to the situation.

Immediate Reinforcement of Skills Following most examples there is the statement, **NOW WORK**. This callout directs students to a related exercise from the section's problem set. Math is best learned actively. Doing a related problem immediately after working through a solved example not only keeps students engaged, but also enhances understanding and strengthens their ability to apply the objective. This practice also serves as a confidence-builder for students.

NOW WORK Problem 35.

The Text Is Written with Ample Opportunity to Practice

Check Comprehension Before the Test Immediately after reading a section, we suggest that students assess their comprehension of the main points of the section by answering the **Concepts and Vocabulary** questions. These are a selection of quick fill in the blank, multiple-choice, and true/false questions. If a student has trouble answering these questions correctly, then a flag is raised immediately—go back and review the section or get help from the instructor.

Concepts and Vocabulary

1. *True or False* $\dfrac{f(x)}{g(x)}$ is an indeterminate form at c of the type $\dfrac{0}{0}$ if $\lim\limits_{x\to c}\dfrac{f(x)}{g(x)}$ does not exist.

Instructors may find Concept and Vocabulary problems useful for a class-opening, for a 5-minute quiz, or for iClicker responses to determine if students are prepared for class.

Skill Building

In Problems 5–18, find each derivative using Part 1 of the Fundamental Theorem of Calculus.

5. $\dfrac{d}{dx}\displaystyle\int_1^x \sqrt{t^2+1}\,dt$ 6. $\dfrac{d}{dx}\displaystyle\int_3^x \frac{t+1}{t}\,dt$

Practice, the Best Way to Learn Calculus The **Skill Building** exercises are grouped into subsets, usually corresponding to the objectives of the section, and are correlated to the solved examples. Working a variety of Skill Building problems increases students' computational skills and ability to choose the best approach to solve a problem. The result: Student success, which builds student confidence.

Applications and Extensions

51. **Uninhibited Growth** The population of a colony of mosquitoes obeys the uninhibited growth equation $\dfrac{dN}{dt} = kN$. If there are 1500 mosquitoes initially, and there are 2500 mosquitoes after 24 h, what is the mosquito population after 3 days?

Having mastered the computational skills, students are ready to tackle the **Application and Extension** problems. These are a diverse set of applied problems, some of which have been contributed by calculus students from various colleges and universities. The Application and Extension section also includes problems that use a slightly different approach to the section material, problems that require proof, and problems that extend the concepts of the section.

Challenge Problems

60. The floor function $f(x) = \lfloor x \rfloor$ is not continuous on $[0, 4]$. Show that $\int_0^4 f(x)\,dx$ exists.

For students requiring more of a challenge, the **Challenge Problems** provide more difficult extensions of the section material. Often combined with concepts learned in previous chapters, Challenge Problems are intended to be thought-provoking problems that require some ingenuity to solve. They are designed to challenge the stronger students in the class.

Chapter Review

THINGS TO KNOW

5.1 Area

Definitions:

- Partition of an interval $[a, b]$ (p. 3)
- Area A under the graph of a function f from a to b (p. 6)

Review, the Way to Reach Higher Levels of Learning Student success can be measured in many ways. Often in school student success is measured by a test. Recognizing this, each chapter, other than Chapter P, concludes with a **Chapter Review.** The review incorporates pedagogical features found in few advanced mathematics texts. Each Chapter Review consists of

- **Things to Know,** a detailed list of important definitions, formulas, and theorems contained in the chapter and page references to where they can be found.
- **Objectives Table,** which follows Things to Know. This table provides a section-by-section list of the learning objectives of the chapter (with page references), along with references to the worked examples and the review problems pertaining to the particular learning objective.

OBJECTIVES

Section	You should be able to …	Examples	Review Exercises
5.1	1 Approximate the area under the graph of a function (p. 344)	1, 2	7, 8
	2 Find the area under the graph of a function (p. 348)	3, 4	9, 10
5.2	1 Define a definite integral as the limit of Riemann sums (p. 353)	1, 2	11(a), (b)

- **Review Problems** that provide a comprehensive review of the key concepts of the chapter, matched to the learning objectives of each section.

By using the review exercises to study, a student can identify the objectives mastered and those that need additional work. By referring back to the objective(s) of problems missed, reviewing the material for that objective, reworking the NOW WORK **Problem(s),** and trying the review problem again, a student should have the skills and the confidence to take an exam or proceed to the next chapter.

Student success is ultimately measured by deep understanding and the ability to extend knowledge. When reviewing and incorporating previous knowledge, students begin to make connections. Once connections are formed, understanding and mastery follow.

In addition to these explicit pedagogical features, there are less obvious, but certainly no less significant, features that have been woven into the text. As educators and as students we realize that learning new material is often made more difficult when the style of presentation changes, so we have striven to keep the language and notation consistent throughout the book. Definitions are used in their entirety, both when presented and later when applied. This consistency is followed throughout the text and the exercises. Such attention to the consistency of language and notation is mirrored in our approach to writing precise mathematics while keeping the language clear and accessible to students.

The pedagogical features are tools that will aid students in their mastery of calculus. But it is the clarity of writing that allows students to master the understanding and to use the tools effectively. The accuracy of the mathematics and transparency in the writing will guarantee that any dedicated student can come to grips with the underlying theories, without having to decipher confusing dialogue.

It is our hope that you can encourage your students to read the textbook, use its features, practice the problems, and get help as soon as they need it. Our desire is that we, as a team, can build a cadre of confident and successful calculus students ready to pursue their dreams and goals!

Organizational Features That Set This Text Apart

Appendix A is a brief review of topics that students might have forgotten, but are used throughout calculus. The content of Appendix A consists of material studied prior to the introduction of functions in precalculus. Although definitions, theorems, and examples are provided in Appendix A, this appendix includes no exercises. The purpose of Appendix A is to refresh the student's memory of previously mastered concepts from prerequisite courses. You may wish to look at the content found in Appendix A. In particular, notice that summation notation, used in Chapter 5, is discussed in Section A.5

Chapter P deals with functions encountered in precalculus. It is designed either to be a quickly covered refresher at the beginning of the course or to be a just-in-time review used when needed. As such, the sections of Chapter P are lean and include only a limited number of practice problems that reinforce the reviewed topics.

Chapter 1 is dedicated to limits and continuity. Although most of the chapter addresses the idea of a limit and methods of finding limits, we state the formal ε-δ definition of limit in Section 1. We also include an example that investigates how close x must be to c to ensure that the difference between $f(x)$ and L is less than some prescribed number. But it is not until Section 6, after students are comfortable with the concept of a limit, the procedures of finding limits, and properties of limits, that the ε-δ definition of limit is repeated, discussed, and applied.

The Derivative has been split into two chapters (Chapters 2 and 3). This allows us to expand the coverage of the derivative and avoid a chapter of unwieldy length. Chapter 2 includes rates of change, the derivative as a function, and the Sum, Difference, Product, Quotient, and Simple Power rules for finding derivatives. It also contains derivatives of the exponential function and the trigonometric functions.

Chapter 3 covers the Chain Rule, implicit differentiation, the derivative of logarithmic and hyperbolic functions, differentials and linear approximations, and the approximation of zeros of functions using Newton's method.

Notice that there is also a section on Taylor polynomials (Section 3.5). We feel that the logical place for Taylor polynomials is immediately after differentials and linear approximations. It is natural to ask, "If a linear approximation to a function f near $x = x_0$ is good in a small interval surrounding x_0, then does a higher degree polynomial provide a better approximation of a function f over a wider interval?"

We begin Chapter 6, by finding the volume of a solid of revolution using the disk and washer method, followed by the shell method. We use both methods to solve several examples. This parallel approach enhances the student's appreciation of which method to choose when asked to find a volume. Only then do we introduce the slicing method, which is a generalization of the disk and washer method.

In Chapter 8, students are asked to recall the Taylor Polynomials studied earlier. The early exposure to the Taylor Polynomials serves two purposes here: First, it distinguishes a Taylor Polynomial from a Taylor Series; and second, it helps to make the idea of convergence of a Taylor Series easier to understand.

In addition to Chapter 16, Differential Equations, we provide examples of differential equations throughout the text. They are first introduced in Chapter 4: Applications of the Derivative, along with antiderivatives (Section 4.8). Again, the placement is a logical consequence of the relationship between derivatives and antiderivatives. At this point the idea of boundary values is introduced. Differential equations are revisited a second time in Chapter 5, The Integral, with the discussion of exponential growth (Section 5.5) and Newton's Law of Cooling (Section 5.6).

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Ensuring the mathematical rigor, accuracy, and precision, as well as the complete coverage and clear language that we have strived for with this text, requires not only a great deal of time and effort on our part, but also on the part of an entire team of exceptional reviewers. First and foremost, Tony Palermino has provided exceptional editorial advice throughout the long development process. Mark Grinshpon has been invaluable for his mathematical insights, analytical expertise, and sheer dedication to the project; without his contributions, the text would be greatly diminished. The following people have also provided immeasurable support reading drafts of chapters and offering insight and advice critical to the success of this textbook:

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Applications Index

Note: *Italics* indicates Example.

P

Preparing for Calculus

Until now, the mathematics you have encountered has centered mainly on algebra, geometry, and trigonometry. These subjects have a long history, well over 2000 years. But calculus is relatively new; it was developed less than 400 years ago.

Calculus deals with change and how the change in one quantity affects other quantities. Fundamental to these ideas are functions and their properties.

In Chapter P, we discuss many of the functions used in calculus. We also provide a review of techniques from precalculus used to obtain the graphs of functions and to transform known functions into new functions.

Your instructor may choose to cover all or part of the chapter. Regardless, throughout the text, you will see the **NEED TO REVIEW?** marginal notes. They reference specific topics, often discussed in Chapter P.

P.1 Functions and Their Graphs

OBJECTIVES *When you finish this section, you should be able to:*

1. Evaluate a function (p. 3)
2. Find the domain of a function (p. 4)
3. Identify the graph of a function (p. 5)
4. Analyze a piecewise-defined function (p. 7)
5. Obtain information from or about the graph of a function (p. 7)
6. Use properties of functions (p. 9)
7. Find the average rate of change of a function (p. 11)

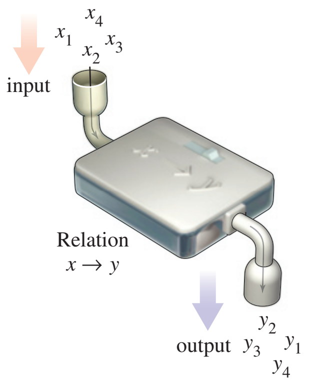

Figure 1

Often there are situations where one variable is somehow linked to another variable. For example, the price of a gallon of gas is linked to the price of a barrel of oil. A person can be associated to her telephone number(s). The volume V of a sphere depends on its radius R. The force F exerted by an object corresponds to its acceleration a. These are examples of a **relation**, a correspondence between two sets called the **domain** and the **range**. If x is an element of the domain and y is an element of the range, and if a relation exists from x to y, then we say that y **corresponds** to x or that y **depends on** x, and we write $x \to y$. It is often helpful to think of x as the **input** and y as the **output** of the relation. See Figure 1.

Suppose an astronaut standing on the moon throws a rock 20 meters up and starts a stopwatch as the rock begins to fall back down. If x represents the number of seconds on the stopwatch and if y represents the altitude of the rock at that time, then there is a relation between time x and altitude y. If the altitude of the rock is measured at $x = 1$, 2, 2.5, 3, 4, and 5 seconds, then the altitude is approximately $y = 19.2$, 16.8, 15, 12.8, 7.2, and 0 meters, respectively.

The astronaut could express this relation numerically, graphically, or algebraically. The relation can be expressed by a table of numbers (see Table 1) or by the set of ordered pairs $\{(0, 20), (1, 19.2), (2, 16.8), (2.5, 15), (3, 12.8), (4, 7.2), (5, 0)\}$, where the first element of each pair denotes the time x and the second element denotes the altitude y. The relation also can be expressed visually, using either a graph, as in Figure 2, or a map, as in Figure 3. Finally, the relation can be expressed algebraically using the formula

$$y = 20 - 0.8x^2$$

TABLE 1

Time, x (in seconds)	Altitude, y (in meters)
0	20
1	19.2
2	16.8
2.5	15
3	12.8
4	7.2
5	0

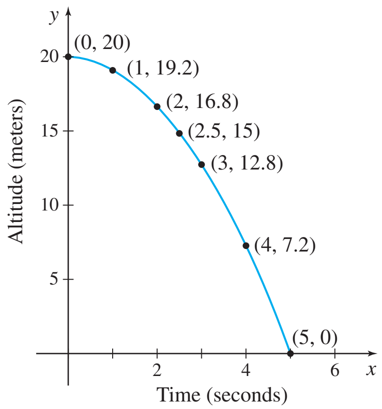

Figure 2

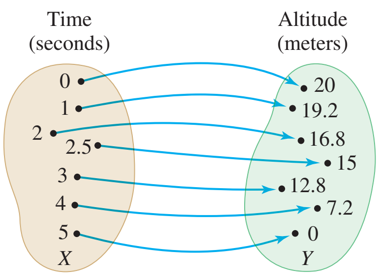

Figure 3

NOTE Not every relation is a function. If any element x in the set X corresponds to more than one element y in the set Y, then the relation is not a function.

In this example, notice that if X is the set of times from 0 to 5 seconds and Y is the set of altitudes from 0 to 20 meters, then each element of X corresponds to one and only one element of Y. Each given time value yields a **unique**, that is, exactly one, altitude value. Any relation with this property is called a *function from X into Y*.

DEFINITION Function

Let X and Y be two nonempty sets.* A **function** f from X into Y is a relation that associates with each element of X exactly one element of Y.

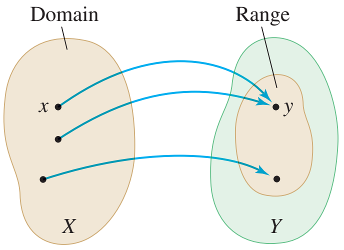

Figure 4

The set X is called the **domain** of the function. For each element x in X, the corresponding element y in Y is called the **value** of the function at x, or the **image** of x. The set of all the images of the elements in the domain is called the **range** of the function. Since there may be elements in Y that are not images of any x in X, the range of a function is a subset of Y. See Figure 4.

1 Evaluate a Function

Functions are often denoted by letters such as f, F, g, and so on. If f is a function, then for each element x in the domain, the corresponding image in the range is denoted by the symbol $f(x)$, read "f of x." $f(x)$ is called the **value of f at x**. The variable x is called the **independent variable** or the **argument** because it can be assigned any element from the domain, while the variable y is called the **dependent variable**, because its value depends on x.

EXAMPLE 1 Evaluating a Function

For the function f defined by $f(x) = 2x^2 - 3x$, find:

(a) $f(5)$ **(b)** $f(x+h)$ **(c)** $f(x+h) - f(x)$ **(d)** $\dfrac{f(x+h) - f(x)}{h}$, $h \neq 0$

Solution (a) $f(5) = 2(5)^2 - 3(5) = 50 - 15 = 35$

(b) The function $f(x) = 2x^2 - 3x$ gives us a rule to follow. To find $f(x+h)$, expand $(x+h)^2$, multiply the result by 2, and then subtract the product of 3 and $(x+h)$.

$$f(x+h) \underset{\substack{\uparrow \\ \text{In } f(x) \text{ replace } x \text{ by } x+h}}{=} 2(x+h)^2 - 3(x+h) = 2(x^2 + 2hx + h^2) - 3x - 3h$$

$$= 2x^2 + 4hx + 2h^2 - 3x - 3h$$

(c) $f(x+h) - f(x) = [2x^2 + 4hx + 2h^2 - 3x - 3h] - [2x^2 - 3x] = 4hx + 2h^2 - 3h$

(d) $$\frac{f(x+h) - f(x)}{h} = \frac{4hx + 2h^2 - 3h}{h} = \frac{h[4x + 2h - 3]}{h} \underset{\substack{\uparrow \\ h \neq 0;\ \text{divide out } h}}{=} 4x + 2h - 3$$ ■

The expression in (d) is called the **difference quotient** of f. Difference quotients occur frequently in calculus, as we will see in Chapters 2 and 3.

NOW WORK Problems 13 and 23.

EXAMPLE 2 Finding the Amount of Gasoline in a Tank

A Shell station stores its gasoline in an underground tank that is a right circular cylinder lying on its side. The volume V of gasoline in the tank (in gallons) is given by the formula

$$V(h) = 40h^2\sqrt{\frac{96}{h} - 0.608}$$

where h is the height (in inches) of the gasoline as measured on a depth stick. See Figure 5.

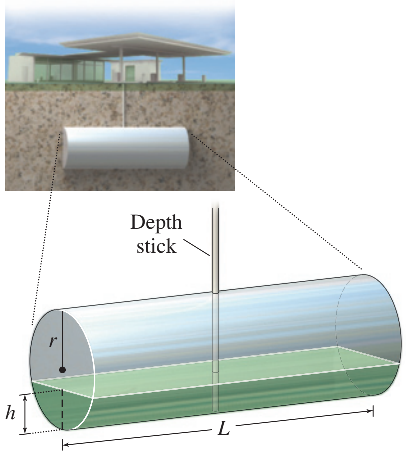

Figure 5

*The sets X and Y will usually be sets of real numbers, defining a **real function**. The two sets could also be sets of complex numbers, defining a **complex function**, or X could be a set of real numbers and Y a set of vectors, defining a **vector-valued function**. In the broad definition, X and Y can be any two sets.

(a) If $h = 12$ inches, how many gallons of gasoline are in the tank?
(b) If $h = 1$ inch, how many gallons of gasoline are in the tank?

Solution (a) We evaluate V when $h = 12$.

$$V(12) = 40(12)^2\sqrt{\frac{96}{12} - 0.608} = 40 \cdot 144\sqrt{8 - 0.608} = 5760\sqrt{7.392} \approx 15{,}660$$

There are about 15,660 gallons of gasoline in the tank when the height of the gasoline in the tank is 12 inches.

(b) Evaluate V when $h = 1$.

$$V(1) = 40(1)^2\sqrt{\frac{96}{1} - 0.608} = 40\sqrt{96 - 0.608} = 40\sqrt{95.392} \approx 391$$

There are about 391 gallons of gasoline in the tank when the height of the gasoline in the tank is 1 inch. ■

Implicit Form of a Function

In general, a function f defined by an equation in x and y is said to be given **implicitly**. If it is possible to solve the equation for y in terms of x, then we write $y = f(x)$ and say the function is given **explicitly**. For example,

Implicit Form	Explicit Form
$x^2 - y = 6$	$y = f(x) = x^2 - 6$
$xy = 4$	$y = g(x) = \frac{4}{x}$

2 Find the Domain of a Function

In applications, the domain of a function is sometimes specified. For example, we might be interested in the population of a city from 1990 to 2012. The domain of the function is time, in years, and is restricted to the interval [1990, 2012]. Other times the domain is restricted by the context of the function itself. For example, the volume V of a sphere, given by the function $V = \frac{4}{3}\pi R^3$, makes sense only if the radius R is greater than 0. But often the domain of a function f is not specified; only the formula defining the function is given. In such cases, the **domain** of f is the largest set of real numbers for which the value $f(x)$ is defined and is a real number.

EXAMPLE 3 Finding the Domain of a Function

Find the domain of each of the following functions:

(a) $f(x) = x^2 + 5x$ **(b)** $g(x) = \dfrac{3x}{x^2 - 4}$

(c) $h(t) = \sqrt{4 - 3t}$ **(d)** $F(u) = \dfrac{5u}{\sqrt{u^2 - 1}}$

Solution (a) Since $f(x) = x^2 + 5x$ is defined for any real number x, the domain of f is the set of all real numbers.

(b) Since division by zero is not defined, $x^2 - 4$ cannot be 0, that is, $x \neq -2$ and $x \neq 2$. The function $g(x) = \dfrac{3x}{x^2 - 4}$ is defined for any real number except $x = -2$ and $x = 2$. So, the domain of g is the set of real numbers $\{x | x \neq -2, x \neq 2\}$.

(c) Since the square root of a negative number is not a real number, the value of $4 - 3t$ must be nonnegative. The solution of the inequality $4 - 3t \geq 0$ is $t \leq \frac{4}{3}$, so the domain of h is the set of real numbers $\left\{t \middle| t \leq \frac{4}{3}\right\}$ or the interval $\left(-\infty, \frac{4}{3}\right]$.

NEED TO REVIEW? Solving inequalities is discussed in Appendix A.1, pp. A-5 to A-8.

(d) Since the square root is in the denominator, the value of $u^2 - 1$ must be not only nonnegative, it also cannot equal zero. That is, $u^2 - 1 > 0$. The solution of the inequality $u^2 - 1 > 0$ is the set of real numbers $\{u|u < -1\} \cup \{u|u > 1\}$ or the set $(-\infty, -1) \cup (1, \infty)$. ■

NEED TO REVIEW? Interval notation is discussed in Appendix A.1, p. A-5.

If x is in the domain of a function f, we say that f **is defined at** x, or $f(x)$ **exists**. If x is not in the domain of f, we say that f **is not defined at** x, or $f(x)$ **does not exist**. The domain of a function is expressed using inequalities, interval notation, set notation, or words, whichever is most convenient. Notice the various ways the domain was expressed in the solution to Example 3.

NOW WORK Problem 17.

3 Identify the Graph of a Function

In applications, often a graph reveals the relationship between two variables more clearly than an equation. For example, Table 2 shows the average price of gasoline at a particular gas station in Texas (for the years 1980–2012 adjusted for inflation, based on 2008 dollars). If we plot these data and then connect the points, we obtain Figure 6.

TABLE 2

Year	Price	Year	Price	Year	Price
1980	3.41	1991	1.90	2002	1.86
1981	3.26	1992	1.82	2003	1.79
1982	3.15	1993	1.70	2004	2.13
1983	2.51	1994	1.85	2005	2.60
1984	2.51	1995	1.68	2006	2.62
1985	2.46	1996	1.87	2007	3.29
1986	1.63	1997	1.65	2008	2.10
1987	1.90	1998	1.50	2009	2.45
1988	1.77	1999	1.73	2010	2.97
1989	1.83	2000	1.85	2011	3.80
1990	2.25	2001	1.40	2012	3.91

Source: http://www.randomuseless.info/gasprice/gasprice.html

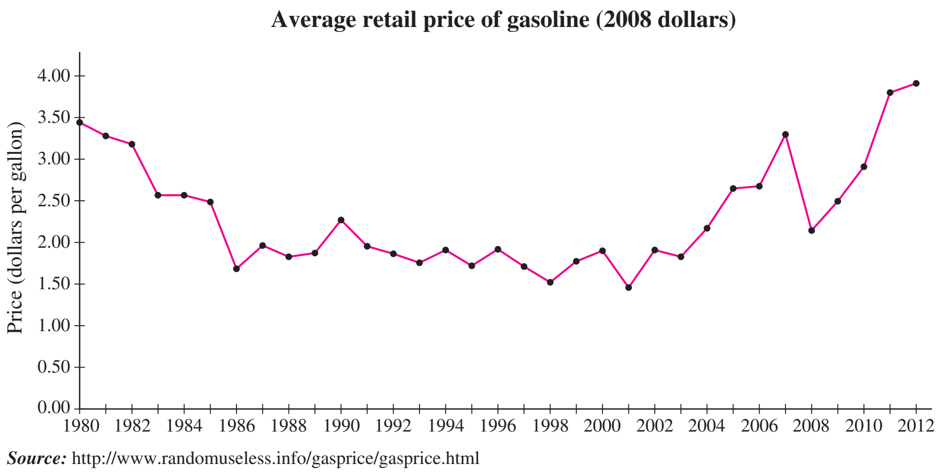

Figure 6

NEED TO REVIEW? The graph of an equation is discussed in Appendix A.3, pp. A-16 to A-20.

The graph shows that for each date on the horizontal axis there is only one price on the vertical axis. So, the graph represents a function, although the rule for determining the price from the year is not given.

When a function is defined by an equation in x and y, the **graph of the function** is the set of points (x, y) in the xy-plane that satisfy the equation.

But not every collection of points in the xy-plane represents the graph of a function. Recall that a relation is a function only if each element x in the domain corresponds to exactly one image y in the range. This means the graph of a function never contains two points with the same x-coordinate and different y-coordinates. Compare the graphs in Figures 7 and 8. In Figure 7 every number x is associated with exactly one number y, but in Figure 8 some numbers x are associated with three numbers y. Figure 7 shows the graph of a function; Figure 8 shows a graph that is not the graph of a function.

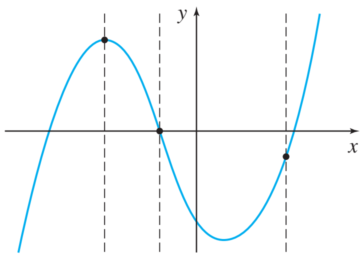

Figure 7 Function: Exactly one y for each x. Every vertical line intersects the graph in at most one point.

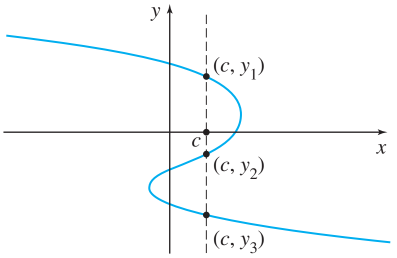

Figure 8 Not a function: $x = c$ has 3 y's associated with it. The vertical line $x = c$ intersects the graph in three points.

For a graph to be a *graph of a function*, it must satisfy the *Vertical-line Test*.

NOTE The phrase "if and only if" means the concepts on each side of the phrase are equivalent. That is, they have the same meaning.

THEOREM Vertical-line Test

A set of points in the xy-plane is the graph of a function if and only if every vertical line intersects the graph in at most one point.

EXAMPLE 4 Identifying the Graph of a Function

Which graphs in Figure 9 represent the graph of a function?

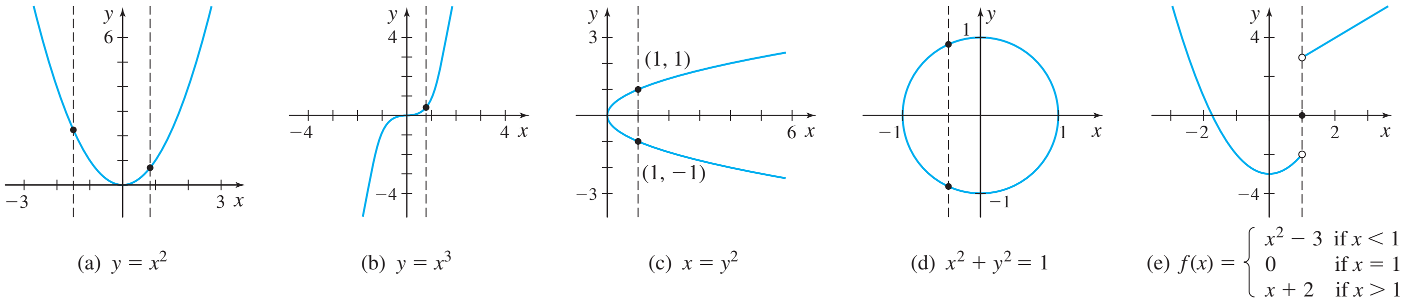

Figure 9

Solution The graphs in Figure 9(a), 9(b), and 9(e) are graphs of functions because every vertical line intersects each graph in at most one point. The graphs in Figure 9(c) and 9(d) are not graphs of functions, because there is a vertical line that intersects each graph in more than one point. ■

NOW WORK Problems 31(a) and (b).

Notice that although the graph in Figure 9(e) represents a function, it looks different from the graphs in (a) and (b). The graph consists of two pieces plus a point and they are not connected. Also notice that different equations describe different pieces of the graph. Functions with graphs similar to the one in Figure 9(e) are called *piecewise-defined functions*.

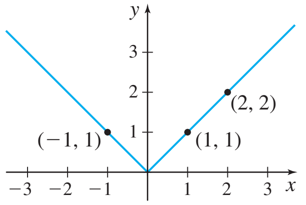

Figure 10 $f(x) = |x|$

RECALL x-intercepts are numbers on the x-axis at which a graph touches or crosses the x-axis.

4 Analyze a Piecewise-Defined Function

Sometimes a function is defined differently on different parts of its domain. For example, the *absolute value function* $f(x) = |x|$ is actually defined by two equations: $f(x) = x$ if $x \geq 0$ and $f(x) = -x$ if $x < 0$. These equations are usually combined into one expression as

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Figure 10 shows the graph of the absolute value function. Notice that the graph of f satisfies the Vertical-line Test.

When a function is defined by different equations on different parts of its domain, it is called a **piecewise-defined** function.

EXAMPLE 5 Analyzing a Piecewise-Defined Function

The function f is defined as

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 10 & \text{if } 0 \leq x \leq 100 \\ 0.2x - 10 & \text{if } x > 100 \end{cases}$$

(a) Evaluate $f(-1)$, $f(100)$, and $f(200)$.
(b) Graph f.
(c) Find the domain, range, and the x- and y-intercepts of f.

Solution (a) $f(-1) = 0$; $f(100) = 10$; $f(200) = 0.2(200) - 10 = 30$

(b) The graph of f consists of three pieces corresponding to each equation in the definition. The graph is the horizontal line $y = 0$ on the interval $(-\infty, 0)$, the horizontal line $y = 10$ on the interval $[0, 100]$, and the line $y = 0.2x - 10$ on the interval $(100, \infty)$, as shown in Figure 11.

(c) f is a piecewise-defined function. Look at the values that x can take on: $x < 0$, $0 \leq x \leq 100$, $x > 100$. We conclude the domain of f is all real numbers. The range of f is the number 0 and all real numbers greater than or equal to 10. The x-intercepts are all the numbers in the interval $(-\infty, 0)$; the y-intercept is 10. ■

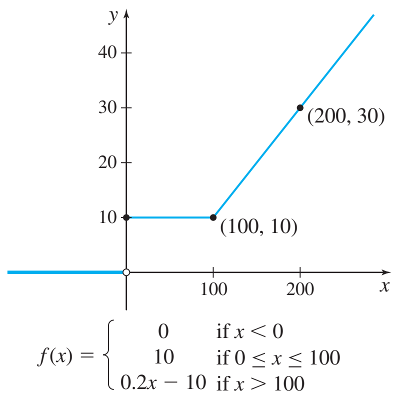

Figure 11

NOW WORK Problem 33.

5 Obtain Information from or about the Graph of a Function

The graph of a function provides a great deal of information about the function. Reading and interpreting graphs is an essential skill for calculus.

EXAMPLE 6 Obtaining Information from the Graph of a Function

The graph of $y = f(x)$ is given in Figure 12. (x might represent time and y might represent the distance of the bob of a pendulum from its *at-rest* position. Negative values of y would indicate that the bob is to the left of its at-rest position; positive values of y would mean that the bob is to the right of its at-rest position.)

(a) What are $f(0)$, $f\left(\frac{3\pi}{2}\right)$, and $f(3\pi)$?
(b) What is the domain of f?
(c) What is the range of f?
(d) List the intercepts of the graph.
(e) How many times does the line $y = 2$ intersect the graph of f?
(f) For what values of x does $f(x) = -4$?
(g) For what values of x is $f(x) > 0$?

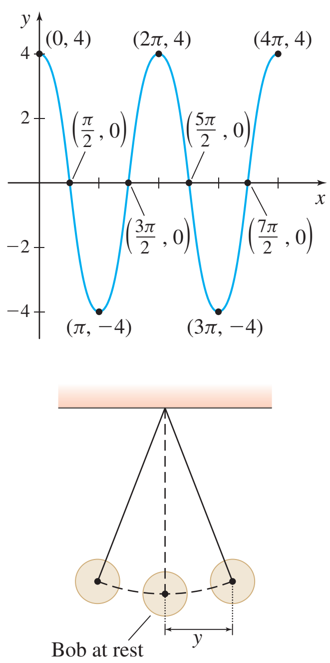

Figure 12

Solution (a) Since the point $(0, 4)$ is on the graph of f, the y-coordinate 4 is the value of f at 0; that is, $f(0) = 4$. Similarly, when $x = \frac{3\pi}{2}$, then $y = 0$, so $f\left(\frac{3\pi}{2}\right) = 0$, and when $x = 3\pi$, then $y = -4$, so $f(3\pi) = -4$.

(b) The points on the graph of f have x-coordinates between 0 and 4π inclusive. The domain of f is $\{x | 0 \le x \le 4\pi\}$ or the closed interval $[0, 4\pi]$.

(c) Every point on the graph of f has a y-coordinate between -4 and 4 inclusive. The range of f is $\{y | -4 \le y \le 4\}$ or the closed interval $[-4, 4]$.

RECALL Intercepts are points at which a graph crosses or touches a coordinate axis.

(d) The intercepts of the graph of f are $(0, 4)$, $\left(\frac{\pi}{2}, 0\right)$, $\left(\frac{3\pi}{2}, 0\right)$, $\left(\frac{5\pi}{2}, 0\right)$, and $\left(\frac{7\pi}{2}, 0\right)$.

(e) Draw the graph of the line $y = 2$ on the same set of coordinate axes as the graph of f. The line intersects the graph of f four times.

(f) Find points on the graph of f for which $y = f(x) = -4$; there are two such points: $(\pi, -4)$ and $(3\pi, -4)$. So $f(x) = -4$ when $x = \pi$ and when $x = 3\pi$.

(g) $f(x) > 0$ when the y-coordinate of a point (x, y) on the graph of f is positive. This occurs when x is in the set $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right) \cup \left(\frac{7\pi}{2}, 4\pi\right]$. ■

NOW WORK Problems 37, 39, 41, 43, 45, 47, and 49.

EXAMPLE 7 Obtaining Information about the Graph of a Function

Consider the function $f(x) = \frac{x+1}{x+2}$.

(a) What is the domain of f?

(b) Is the point $\left(1, \frac{1}{2}\right)$ on the graph of f?

(c) If $x = 2$, what is $f(x)$? What is the corresponding point on the graph of f?

(d) If $f(x) = 2$, what is x? What is the corresponding point on the graph of f?

(e) What are the x-intercepts of the graph of f (if any)? What point(s) on the graph of f correspond(s) to the x-intercept(s)?

Solution (a) The domain of f consists of all real numbers except -2; that is, the set $\{x | x \ne -2\}$.

(b) When $x = 1$, then $f(1) \underset{x=1}{\uparrow}{=} \frac{1+1}{1+2} = \frac{2}{3}$. The point $\left(1, \frac{2}{3}\right)$ is on the graph of f; the point $\left(1, \frac{1}{2}\right)$ is not on the graph of f.

(c) If $x = 2$, then $f(2) \underset{x=2}{\uparrow}{=} \frac{2+1}{2+2} = \frac{3}{4}$. The point $\left(2, \frac{3}{4}\right)$ is on the graph of f.

(d) If $f(x) = 2$, then $\frac{x+1}{x+2} = 2$. Solving for x, we find

$$x + 1 = 2(x + 2) = 2x + 4$$
$$x = -3$$

The point $(-3, 2)$ is on the graph of f.

(e) The x-intercepts of the graph of f occur when $y = 0$. That is, they are the solutions of the equation $f(x) = 0$. The x-intercepts are also called the real **zeros** or **roots** of the function f.

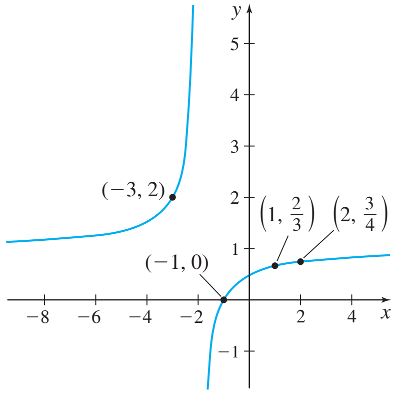

Figure 13 $f(x) = \dfrac{x+1}{x-2}$

The real zeros of the function $f(x) = \dfrac{x+1}{x+2}$ satisfy the equation $x + 1 = 0$ or $x = -1$. The only x-intercept is -1, so the point $(-1, 0)$ is on the graph of f. ■

Figure 13 shows the graph of f.

NOW WORK Problems 55, 57, 59.

6 Use Properties of Functions

One of the goals of calculus is to develop techniques for graphing functions. Here we review some properties of functions that help obtain the graph of a function.

DEFINITION Even and Odd Functions

A function f is **even** if, for every number x in its domain, the number $-x$ is also in the domain and

$$\boxed{f(-x) = f(x)}$$

A function f is **odd** if, for every number x in its domain, the number $-x$ is also in the domain and

$$\boxed{f(-x) = -f(x)}$$

For example, $f(x) = x^2$ is an even function since

$$f(-x) = (-x)^2 = x^2 = f(x)$$

Also, $g(x) = x^3$ is an odd function since

$$g(-x) = (-x)^3 = -x^3 = -g(x)$$

NEED TO REVIEW? Symmetry of equations is discussed in Appendix A.3, pp. A-17 to A-18.

See Figure 14 for the graph of $f(x) = x^2$ and Figure 15 for the graph of $g(x) = x^3$. Notice that the graph of the even function $f(x) = x^2$ is symmetric with respect to the y-axis and the graph of the odd function $g(x) = x^3$ is symmetric with respect to the origin.

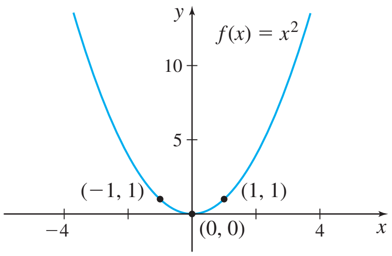

Figure 14 The function $f(x) = x^2$ is even. The graph of f is symmetric with respect to the y-axis.

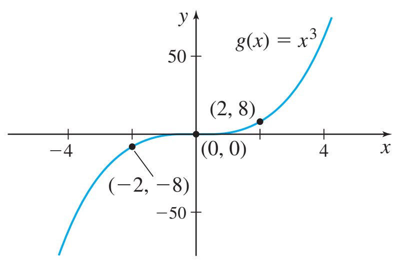

Figure 15 The function $g(x) = x^3$ is odd. The graph of g is symmetric with respect to the origin.

THEOREM Graphs of Even and Odd Functions

- A function is even if and only if its graph is symmetric with respect to the y-axis.
- A function is odd if and only if its graph is symmetric with respect to the origin.

EXAMPLE 8 Identifying Even and Odd Functions

Determine whether each of the following functions is even, odd, or neither. Then determine whether its graph is symmetric with respect to the y-axis, the origin, or neither.

(a) $f(x) = x^2 - 5$ **(b)** $g(x) = \dfrac{4x}{x^2 - 5}$ **(c)** $h(x) = \sqrt[3]{5x^3 - 1}$

(d) $F(x) = |x|$ **(e)** $H(x) = \dfrac{x^2 + 2x - 1}{(x-5)^2}$